

THE MORSE–BOTT–KIRWAN CONDITION IS LOCAL

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ABSTRACT. Kirwan identified a condition on a smooth function under which the usual techniques of Morse–Bott theory can be applied to this function. We prove that if a function satisfies this condition locally then it also satisfies the condition globally. As an application, we use the local normal form theorem to recover Kirwan’s result that the norm-square of a momentum map satisfies Kirwan’s condition.

1. INTRODUCTION

For a Hamiltonian action of a compact Lie group on a compact symplectic manifold, a fundamental work of Frances Kirwan [6] makes it possible to apply Morse theoretic techniques to the norm-square of the momentum map. The norm-square is not a Morse–Bott function; components of its critical set might not even be smooth. Kirwan identifies a condition on a real valued function: being *minimally degenerate*. This condition is more general than the Morse–Bott condition; nevertheless, it allows one to apply the machinery of Morse theory. Nowadays, this condition is sometimes called “Morse–Bott in the sense of Kirwan.”

Applying Morse–theoretic arguments to the norm-square of a momentum map is the main ingredient in the proof of Kirwan surjectivity, a pivotal result in equivariant symplectic geometry and geometric invariant theory, as well as in the study of the global topology of Hamiltonian compact group actions. These techniques played a central role in the confirmation of physicists’ predictions for the structure of the cohomology ring of moduli spaces of holomorphic vector bundles over Riemann surfaces [5].

Kirwan’s definition of a minimally degenerate function is not local: it requires the set of critical points to be a disjoint union of closed subsets, each of which has a neighbourhood that satisfies a certain condition. If the set of critical points is not discrete, then, in contrast to the Morse–Bott condition, *a priori* it is not clear if one can tell that a function is “minimally degenerate” by examining small neighbourhoods of individual critical points. This aspect of the definition makes it difficult to check whether a function satisfies this condition.

The purpose of this paper is to prove that Kirwan’s “minimally degenerate” condition actually *is* a local condition: if a function is minimally degenerate near each critical point, then it is minimally degenerate. As an application we use the local normal form theorem for compact Hamiltonian group actions to re-prove Kirwan’s result that the norm-square of the momentum map is minimally degenerate.

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Section 2 contains the basic definitions and the statement of our main theorem, Theorem 2.6. Section 3 contains three propositions that constitute the proof. One of these propositions is proved in Section 3; Sections 4 and 5 are devoted to the proofs of the other two propositions, thus completing the proof of Theorem 2.6. In Section 6, we give a Morse-type lemma, Lemma 6.1, that characterizes minimally degenerate functions in local coordinates. Then in Section 7 we use Theorem 2.6 and the local normal form theorem to re-prove Kirwan’s result that, for a Hamiltonian action of a compact Lie group, the norm-square of a momentum map is minimally degenerate. Finally, in Section 8, we recall some consequences of this fact.

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2. STATEMENT OF THE MAIN RESULT

Kirwan’s definition of a minimally degenerate function relies on the following notion.

Definition 2.1. Let W be a smooth manifold and $f: W \rightarrow \mathbb{R}$ a smooth function. Let C be a closed subset of W on which f is constant and such that every point of C is a critical point of f . A **minimizing manifold** for C is a submanifold N of W that contains C and that has the following two properties.

- (1) For each point x of C , the tangent space $T_x N$ is maximal among subspaces of $T_x W$ on which the Hessian of f is positive semidefinite.
- (2) The restriction $f|_N$ of f to N attains its minimum at exactly the points of C .

Remark 2.2. For any splitting $TW|_N = TN \oplus E$, Condition (1) above is equivalent to the following condition:

- (1') For each point x of C , the space E_x is maximal among subspaces of $T_x W$ on which the Hessian of f is negative definite.

Throughout this paper, we do not assume that the dimension of a manifold is constant. That is, by “manifold”, we allow a disjoint union of manifolds of different dimensions. In particular, in Definition 2.1, if C is not connected, we allow N to have connected components of different dimensions. Similarly, we allow a “vector bundle” over a manifold W to have different ranks over different components of W .

Definition 2.3. Let W be a smooth manifold and $f: W \rightarrow \mathbb{R}$ a smooth function. We say f is **minimally degenerate** if its critical set is a locally finite disjoint union of closed subsets for which there exist minimizing manifolds.

Remark 2.4. Definition 2.3 was proposed by Kirwan [6, page 6], with two differences. First, Kirwan assumes that the set of critical points is a *finite* disjoint union of closed subsets that admit minimizing submanifolds. Second, she requires that the minimizing submanifolds be co-orientable. As for the first difference, the weaker assumption “locally finite” is sufficient for our purposes; it is equivalent to Kirwan’s

assumption “finite” when the manifold is compact, as is the case for complex projective manifolds, which is the case in which Kirwan was the most interested. As for the second difference, co-orientability is required for many consequences (see Remark 2.5) and is guaranteed in our main application in symplectic geometry; it is not essential, though, when we study the differential topological properties of minimally degenerate functions.

Remark 2.5. Every Morse–Bott function is minimally degenerate with the stable manifolds serving as the minimizing manifolds. What Kirwan showed is that one can apply the usual techniques of Morse–Bott theory to a minimally degenerate function even if the function is not a Morse–Bott function.

The usual definition of a Morse–Bott function does not require the negative normal bundle to a critical set to be orientable. However, to obtain Morse–Bott inequalities for ranks of cohomology groups with coefficients in a field \mathbb{F} , the negative normal bundles must be \mathbb{F} –**orientable**. Note that every vector bundle is $\mathbb{Z}/2\mathbb{Z}$ –orientable, and that orientability is equivalent to \mathbb{Q} –orientability. For further details, see [10, §2.6].

Just as for Morse–Bott functions, to Definition 2.3 we may add the requirement that the normal bundles to the minimizing manifolds be \mathbb{F} –orientable. Conclusions in cohomology must then be adjusted to have \mathbb{F} coefficients.

Our main result is that the existence of minimizing manifolds can be checked locally. Co-orientability of the minimizing manifolds is inherently a global property; nevertheless, we also give a condition that guarantees co-orientability and that can be checked locally.

Theorem 2.6. *Let M be a smooth manifold and $f: M \rightarrow \mathbb{R}$ a smooth function. Suppose that every point in M has an open neighbourhood U such that $f|_U$ is minimally degenerate. Then f is minimally degenerate.*

Suppose in addition that there exists an almost complex structure J on M such that at every critical point of f , the Hessian of f is J –invariant. Then f is minimally degenerate with co-orientable minimizing submanifolds.

Tolman and Weitsman note in [12, p. 759] that minimally degenerate functions “morally ... look like the product of a minimum and a non-degenerate Morse–Bott function”. In Theorem 6.1 we use our locality result, Theorem 2.6, to make this description rigorous. Namely, we prove the equivalent statement, that a minimally degenerate function is locally the product of a minimum and a non-degenerate maximum.

3. OUTLINE OF THE PROOF

In this section we give three propositions that constitute the proof of Theorem 2.6. We begin with a technical lemma:

Lemma 3.1. *Let M be a smooth manifold and $f: M \rightarrow \mathbb{R}$ a smooth function. Suppose that every point in M has an open neighbourhood U such that $f|_U$ is minimally degenerate. Then the critical set of f is a locally finite disjoint union of closed subsets on which f is constant.*

Proof. The definition of “minimally degenerate” implies that each point has a neighbourhood in which the function takes only finitely many critical values. The conclusion of the lemma then holds when we take the closed subsets to be the intersections $f^{-1}(c_i) \cap \text{Crit } f$ where c_i are the critical values of f . \square

To prove Theorem 2.6, we fix a manifold M and a smooth function $f: M \rightarrow \mathbb{R}$ that is locally minimally degenerate. Lemma 3.1 gives a decomposition of the critical set of f into a locally finite disjoint union of closed subsets on which f is constant. We may now focus on one such a subset; call it C . Thus, C is closed in M and has a neighbourhood whose intersection with the critical set of f is exactly C , and f is constant on C . We need to prove that there exists a minimizing submanifold for C . We will do this in three steps, formulated below as Propositions 3.3, 3.4, and 3.5.

We will use the following notions. Given a submersion $\pi: U \rightarrow \mathcal{O}$, its **vertical tangent space** at a point x of U is $\ker d\pi|_x$; the point x is a **fibrewise critical point** of a smooth function $g: U \rightarrow \mathbb{R}$ if $dg|_x$ vanishes on the vertical tangent space at x ; and U is **fibrewise orientable** if its vertical tangent bundle is an orientable vector bundle.

In Propositions 3.4 and 3.5 (and only there) we will use the following notion, which applies to the function $f: M \rightarrow \mathbb{R}$ and the critical set C of our setup.

Definition 3.2. A **compatibly fibrated neighbourhood** of C is a neighbourhood U of C together with a submersion $\pi: U \rightarrow \mathcal{O}$ such that at every point x in C the vertical tangent space at x is maximal among subspaces of $T_x M$ on which the Hessian of f is negative definite.

We will informally refer to a submanifold that satisfies Condition (1) but not necessarily Condition (2) of Definition 2.1 as an **infinitesimally minimizing submanifold** for C and to a function whose critical set is a locally finite disjoint union of closed subsets for which there exist infinitesimally minimizing submanifolds as an **infinitesimally minimally degenerate** function.

The following three propositions complete our proof of Theorem 2.6.

Proposition 3.3 (Infinitesimal minimal degeneracy). *Suppose that every point x of C has an open neighbourhood U_x such that there exists a minimizing submanifold for $C \cap U_x$ in U_x . Then there exists a submanifold N of M that contains C and that satisfies Condition (1) of Definition 2.1.*

Suppose in addition that there exists an almost complex structure J on M such that at every point of C the Hessian of f is J -invariant. Then there exists such an N that is co-orientable.

Proposition 3.4 (Compatibly fibrated neighbourhood). *Suppose that there exists a submanifold N of M that contains C and that satisfies Condition (1) of Definition 2.1. Then C has a compatibly fibrated neighbourhood.*

Suppose in addition that N is co-orientable. Then C has a compatibly fibrated neighbourhood that is fibrewise orientable.

Proposition 3.5 (Minimal degeneracy). *Let \tilde{N} denote the set of fibrewise critical points of a compatibly fibrated neighbourhood of C . Then, after possibly intersecting with a smaller neighbourhood of C , the following is true.*

- (a) \tilde{N} is a submanifold of M that contains C and that satisfies Condition (1) of Definition 2.1.

If in addition the compatibly fibrated neighbourhood of C is orientable, then \tilde{N} is co-orientable.

- (b) *Suppose that every point x of C has an open neighbourhood U_x such that there exists a minimizing submanifold for $C \cap U_x$ in U_x . Then \tilde{N} also satisfies Condition (2) of Definition 2.1. Thus, \tilde{N} is a minimizing submanifold for C .*

We prove Proposition 3.3 in Section 4 and Proposition 3.5 in Section 5. We close this section with a proof of Proposition 3.4.

Proof of Proposition 3.4. Let N be a submanifold of M that contains C and that satisfies Condition (1) of Definition 2.1. Namely, for each point x of C , the tangent space $T_x N$ is maximal among subspaces of $T_x M$ on which the Hessian of f is positive semidefinite.

Let $\pi: U \rightarrow N$ be a tubular neighbourhood for N . That is, U is an open neighbourhood of N in M , and there exists a diffeomorphism between U and a disc sub-bundle of the normal bundle to N in M that carries the map π to the projection map in the normal bundle. In particular, for each $x \in N$, the tangent to the fibre of π at x is complementary to $T_x N$ in $T_x M$.

Here, N serves two roles: it is a submanifold of M that contains C , and it is the basis of the fibration π . To match the notation of Definition 3.2, we use a different symbol for the basis of the fibration. Namely, we write \mathcal{O} instead of N for the basis of the fibration, so that the tubular neighbourhood map becomes a map $\pi: U \rightarrow \mathcal{O}$.

By the equivalence of the Conditions (1) and (1') in Definition 2.1 and Remark 2.2, for each $x \in C$, the tangent to the fibre of π at x is maximal among subspaces of $T_x M$ on which the Hessian of f is negative definite. So $\pi: U \rightarrow \mathcal{O}$ is a compatibly fibrating neighbourhood of C .

Finally, we consider the case when N is co-orientable. An orientation on the fibres of the normal bundle of N in M restricts to an orientation on the fibres of the disc sub-bundle, and, in turn, gives an orientation on the fibres of the compatibly fibrating neighbourhood that we have constructed. Thus, the compatibly fibrating neighbourhood is fibrewise orientable, as desired. \square

4. INFINITESIMAL MINIMAL DEGENERACY

Let M be a smooth manifold and $f: M \rightarrow \mathbb{R}$ a smooth function. Let C be a closed subset of the set of critical points of f on which f is constant and that has a neighbourhood whose intersection with the set of critical points of f is exactly C . We assume that every point x of C has an open neighbourhood U_x and a submanifold $N_x \subset U_x$ that contains $C \cap U_x$ such that, for each $x' \in C \cap U_x$, the subspace $T_{x'} N_x$ is maximal among subspaces of $T_{x'} M$ on which the Hessian $\text{Hess } f|_{x'}$ is positive semidefinite.

The purpose of this section is to prove Proposition 3.3. The above hypotheses are slightly weaker than those of Proposition 3.3, in that we are only invoking Condition (1) of the definition of “minimizing submanifold”. It remains for us to prove the conclusion of this proposition:

There exists a submanifold N of M that contains C such that, for each $x' \in C$, the subspace $T_{x'} N$ is maximal among subspaces of $T_{x'} M$ on which the Hessian $\text{Hess } f|_{x'}$ is positive semidefinite.

If in addition there exists an almost complex structure J on M such that at every point of C , the Hessian of f is J -invariant, then there exists such an N that is co-orientable.

Thus, we are making the local assumption that near each point x of C there exists a submanifold N_x that is “infinitesimally minimizing” in the sense that it satisfies Condition (1) of Definition 2.1, and we would like to obtain the global conclusion, that an “infinitesimally minimizing” submanifold N exists globally, near all of C .

The challenge is to “patch together” the submanifolds N_x to form a manifold N . A priori it is not clear how to “patch together” submanifolds. We can “patch together” functions, by means of a partition of unity, so our first attempt is to express each N_x as the regular zero set of a function $h_x: U_x \rightarrow \mathbb{R}^k$ and to take $h := \sum \rho_i h_{x_i}$, where $\{\rho_i\}$ is a partition of unity on the union of the sets U_x with $\text{supp } \rho_i \subset U_{x_i}$. To guarantee that zero remains a regular value of h near C , we require that the differentials of h_{x_i} and h_{x_j} coincide at the points of $C \cap U_{x_i} \cap U_{x_j}$. If this can be arranged then

$$\{h = 0\} \cap \{\text{an appropriate neighbourhood of } C\}$$

is a minimizing submanifold for C .

One problem with this approach is that a regular level set of a function to \mathbb{R}^k must have a trivial normal bundle. So this method already cannot work in the case that f is a Morse–Bott function and the negative normal bundle of C is nontrivial. To fix this, instead of working with functions to \mathbb{R}^k , we work with sections of a rank k vector bundle E . We carry out this plan in the following three lemmas.

Lemma 4.1. *There exists a neighbourhood U of C , and a subbundle E of $TM|_U$, such that at each point x of C the subspace E_x of $T_x M$ is maximal among subspaces on which $\text{Hess } f|_x$ is negative definite.*

Moreover, let J be an almost complex structure on M such that at every point of C the Hessian $\text{Hess } f|_x$ is J -invariant. Then the bundle E may be chosen to be complex, and hence orientable.

Proof. We will construct such a bundle E as a sum of eigenbundles of a fibrewise automorphism $A: TM \rightarrow TM$, near C .

First, we will extend the Hessian of f , which is only defined at critical points, to a symmetric 2-tensor B that is defined on all of M . For this, let $\{U_\alpha\}$ be domains of coordinate charts that cover M ; let B_α be the symmetric 2-tensor on U_α that in the local coordinates on U_α is represented by the matrix of second partial derivatives of f ; and take $B = \sum \rho_\alpha B_\alpha$, where $\{\rho_\alpha: M \rightarrow \mathbb{R}\}$ is a partition of unity with $\text{supp } \rho_\alpha \subset U_\alpha$.

Next, choose a Riemannian metric $\langle \cdot, \cdot \rangle$ on M , and define $A: TM \rightarrow TM$ by $B(u, v) = \langle u, Av \rangle$. Because $B(\cdot, \cdot)$ is symmetric, A is self adjoint with respect to $\langle \cdot, \cdot \rangle$, and so A is diagonalizable. For each $x' \in M$, let $\lambda_{1,x'}, \dots, \lambda_{n,x'}$ denote the eigenvalues of $A|_{x'}$, in increasing order. For each i , the eigenvalue $\lambda_{i,x'}$ is continuous, but perhaps not smooth, as a function of x' .

The assumption of the lemma implies that for every $x \in C$ there exists a neighbourhood U_x and an integer k_x (namely, the codimension of N_x) such that, for every $x' \in U_x \cap C$, the automorphism A of $T_{x'} M$ has exactly k_x negative eigenvalues.

Let C_k be the closed subset of C where $k_x = k$. Then there exists a neighbourhood U_k of C_k such that for all $x' \in U$ the eigenvalue $\lambda_{k+1,x'}$ is strictly

greater than the eigenvalues $\lambda_{1,x'}, \dots, \lambda_{k,x'}$. Shrink the sets U_k so that their closures become disjoint. For each $x' \in U_k$, let $E_{x'}$ be the sum of the eigenspaces of $A|_{x'}: T_{x'}M \rightarrow T_{x'}M$ that correspond to the eigenvalues $\lambda_{1,x'}, \dots, \lambda_{k,x'}$. Then E is a smooth subbundle of $TM|_{\bigcup_k U_k}$, and at each point x of C , E_x is a maximal subspace of T_xM on which $\text{Hess } f|_x$ is negative definite.

Finally, suppose that the Hessian is J -invariant. By averaging, we can arrange the tensor B and the Riemannian metric $\langle \cdot, \cdot \rangle$ to be J -invariant as well. The automorphism $A: TM \rightarrow TM$ is then complex linear, so its eigenbundles are J -invariant, and E is a complex, and hence orientable, vector bundle. \square

Let $h: U \rightarrow E|_U$ be a smooth section of a vector bundle E , and let $x \in U$ be a point where this section vanishes. The **vertical differential** of h at x is the composition of the differential $dh|_x: T_xM \rightarrow T_{(x,0)}E$ with the projection to the second factor in the decomposition

$$\begin{aligned} T_{(x,0)}E &\cong T_{(x,0)}(\text{the zero section of } E) \oplus T_{(x,0)}(\text{the fiber } E_x \text{ of } E) \\ &\cong T_xM \oplus E_x. \end{aligned}$$

Lemma 4.2. *Let U be a neighbourhood of C and E a subbundle of $TM|_U$, such that at each point x of C the subspace E_x of T_xM is maximal among subspaces on which $\text{Hess } f|_x$ is negative definite.*

Then, after possibly shrinking U to a smaller neighbourhood of C , there exists a smooth section $h: U \rightarrow E|_U$ that vanishes on C and such that at each critical point $x \in C$ the restriction to $E|_x$ of the vertical differential of h is the identity map on E_x .

Proof. For each $x \in C$ let U_x be a neighbourhood of x and N_x a submanifold of U_x such that for every $x' \in C \cap U_x$ the subspace $T_{x'}N_x$ of $T_{x'}M$ is maximal among subspaces on which $\text{Hess } f|_{x'}$ is positive semidefinite.

Let $x \in C$. After possibly shrinking U_x , we choose coordinates on U_x ,

$$\varphi: U_x \rightarrow \mathbb{R}^n,$$

in which the submanifold N_x is given by the equations $\varphi_1 = \dots \varphi_k = 0$. Moreover, after possibly shrinking U_x further, we assume that U_x is contained in U . Because T_xN_x is complementary to E_x , after possibly shrinking U_x yet further, the differentials of $\varphi_1, \dots, \varphi_k$ give a trivialization of $E|_{U_x}$:

$$E|_{U_x} \rightarrow \mathbb{R}^k.$$

Let

$$h_x: U_x \rightarrow E|_{U_x}$$

be the section whose composition with the trivialization $E|_{U_x} \rightarrow \mathbb{R}^k$ is the map $(\varphi_1, \dots, \varphi_k)$. Then h_x vanishes on $C \cap U_x$, and, at each $x' \in C \cap U_x$, the restriction to $E_{x'}$ of the differential of h_x is the identity map on $E_{x'}$.

After shrinking U , assume that $U = \bigcup_{x \in C} U_x$. Define a section $h: U \rightarrow E$ by

$$h = \sum_{\alpha} \rho_{\alpha} h_{x_{\alpha}}$$

where $\{\rho_{\alpha}: U \rightarrow \mathbb{R}\}$ is a partition of unity with $\text{supp } \rho_{\alpha} \subset U_{x_{\alpha}}$. Then h satisfies the required properties.

Indeed, let $x' \in C \cap U$. Then $h(x') = \sum_{\alpha} \rho_{\alpha} h_{x_{\alpha}}(x')$. Since $h_{x_{\alpha}}(x') = 0$ for each α , we get that $h(x') = 0$. Choosing a local trivialization of E near x' , and identifying sections of E with \mathbb{R}^k valued functions, for every $v \in T_{x'}M$ we have

$$dh|_{x'}(v) = \sum_{\alpha} \rho_{\alpha} dh_{x_{\alpha}}|_{x'}(v) + h_{x_{\alpha}}(x') d\rho_{\alpha}|_{x'}(v).$$

Because $h_{x_{\alpha}}(x') = 0$ for every α , this expression becomes $\sum_{\alpha} \rho_{\alpha} dh_{x_{\alpha}}|_{x'}(v)$. Let $\bar{v} = (v_1, \dots, v_k)$ be the expression of v with respect to the chosen coordinates. If $v \in E_{x'}$, then $dh_{x_{\alpha}}|_{x'}(v) = \bar{v}$ for each α ; this implies that $dh_{x'}(v) = \bar{v}$, as required. \square

Lemma 4.3. *Let U be a neighbourhood of C and E a subbundle of $TM|_U$, such that at each point x of C the subspace E_x of T_xM is maximal among subspaces on which $\text{Hess } f|_x$ is negative definite.*

Let $h: U \rightarrow E|_U$ be a smooth section that vanishes on C and such that at each critical point $x \in C$ the restriction to E_x of the vertical differential of h is the identity map on E_x .

Let $N_C = h^{-1}(0)$. Then, after possibly shrinking the neighbourhood U of C , the set N_C is a submanifold of U , and for every $x \in C$ the subspace T_xN_C of T_xM is maximal among those subspaces on which $\text{Hess } f|_x$ is positive semidefinite.

If in addition, E is an orientable vector bundle, then N_C is co-orientable.

Proof. Fix a point $x \in C$, and trivialize E in a neighbourhood U_x of x . In terms of this trivialization, h becomes a map from U_x to \mathbb{R}^k . The hypotheses guarantee that the differential of h at x is onto. By the implicit function theorem, after possibly shrinking U_x , the intersection $U_x \cap h^{-1}(0)$ is a submanifold of U_x . So we have shown that every point in C has a neighbourhood U_x whose intersection with $N_C = h^{-1}(0)$ is a submanifold of U_x . After possibly shrinking U , we obtain that N_C itself is a submanifold of U .

Next, we claim that for each $x \in C$ the subspace T_xN_C of T_xM is maximal among subspaces on which $\text{Hess } f|_x$ is positive semidefinite. Indeed, by the implicit function theorem the tangent T_xN_C is the kernel of the vertical differential of h ; by the construction of h the vertical differential of h is a projection to E_x , so its kernel is complementary to E_x . Since E_x is maximal among subspaces of T_xM on which $\text{Hess } f|_x$ is negative definite, every complementary subspace is maximal among subspaces of T_xM on which $\text{Hess } f|_x$ is positive semidefinite.

Finally, we note that if E is an orientable vector bundle, then N_C is co-orientable. Indeed, we have constructed N_C so that its normal bundle is the pullback of E , and co-orientability is precisely orientability of the normal bundle. \square

5. MINIMAL DEGENERACY

Let M be a smooth manifold and $f: M \rightarrow \mathbb{R}$ a smooth function. Let C be a closed subset of the set of critical points of f on which f is constant and that has a neighbourhood whose intersection with the set of critical points of f is exactly C . The purpose of this section is to prove Proposition 3.5. We recall its statement:

Let \tilde{N} denote the set of fibrewise critical points of a compatibly fibrated neighbourhood of C . Then, after possibly intersecting with a smaller neighbourhood of C , the following is true.

- (a) \tilde{N} is a submanifold of M that contains C and that satisfies Condition (1) of Definition 2.1.

If, in addition, the compatibly fibrated neighbourhood of C is fibrewise orientable, then \tilde{N} is co-oriented.

- (b) Suppose that every point x of C has an open neighbourhood U_x such that, for the function $f|_{U_x}: U_x \rightarrow \mathbb{R}$, there exists a minimizing submanifold for $C \cap U_x$. Then \tilde{N} also satisfies Condition (2) of Definition 2.1.

Thus, \tilde{N} is a minimizing submanifold for C .

We now proceed to prove this result. Let $\pi: U \rightarrow \mathcal{O}$ be a compatibly fibrated neighbourhood of C . Namely, U is a neighbourhood of C in M , π is a submersion, and at every point x in C the vertical tangent space at x is maximal among subspaces of $T_x M$ on which the Hessian of f is negative definite. Let \tilde{N} denote the set of fibrewise critical points of π , that is, the points whose vertical tangent space is in the kernel of df .

Lemma 5.1. *For every point x of C there exists a neighbourhood W_x such that the intersection $N_{W_x} := W_x \cap \tilde{N}$ has the following properties.*

- N_{W_x} is a manifold.
- At every point x' of $N_{W_x} \cap C$, the vertical tangent space $\ker d\pi|_{x'}$ is complementary to $T_{x'} N_{W_x}$ in $T_{x'} M$.

Proof. Because π is a submersion, without loss of generality we may identify a neighbourhood of x in M with a neighbourhood of the origin in $\mathbb{R}^a \times \mathbb{R}^b$, such that x becomes the origin, and such that π becomes the projection map

$$\pi(\xi_1, \dots, \xi_a, \eta_1, \dots, \eta_b) = (\eta_1, \dots, \eta_b).$$

The vertical differential of f then becomes the function

$$d_V f: \mathbb{R}^a \times \mathbb{R}^b \rightarrow \mathbb{R}^b$$

that is given by

$$\left(\frac{\partial f}{\partial \eta_1}, \dots, \frac{\partial f}{\partial \eta_b} \right).$$

The set \tilde{N} of fibrewise critical points is precisely the zero set of $d_V f$. By the Implicit Function Theorem, to show that \tilde{N} is a manifold near x , it is enough to show that the differential of $d_x f$ at the origin,

$$(5.2) \quad d(d_V f)|_0: \mathbb{R}^a \times \mathbb{R}^b \rightarrow \mathbb{R}^b$$

is onto; and to show that the tangent space $T_x \tilde{N}$ is transverse to the fibres of π , it is enough to show that the kernel of (5.2) is transverse to the subspace $\mathbb{R}^a \times \{0\}$ of $\mathbb{R}^a \times \mathbb{R}^b$. In coordinates, the linear map (5.2) is represented by the $(a+b) \times b$ matrix

$$\begin{pmatrix} \frac{\partial^2 f}{\partial \xi_1 \partial \eta_1} \Big|_0 & \cdots & \frac{\partial^2 f}{\partial \xi_a \partial \eta_1} \Big|_0 & \frac{\partial^2 f}{\partial^2 \eta_1} \Big|_0 & \cdots & \frac{\partial^2 f}{\partial \eta_b \partial \eta_1} \Big|_0 \\ \vdots & & \vdots & \vdots & & \vdots \\ \frac{\partial^2 f}{\partial \xi_1 \partial \eta_b} \Big|_0 & \cdots & \frac{\partial^2 f}{\partial \xi_a \partial \eta_b} \Big|_0 & \frac{\partial^2 f}{\partial \eta_1 \partial \eta_b} \Big|_0 & \cdots & \frac{\partial^2 f}{\partial^2 \eta_b} \Big|_0 \end{pmatrix}.$$

The above facts that we need both follow from that, by our construction of the coordinates $\mathbb{R}^a \times \mathbb{R}^b$, the right $b \times b$ block of this matrix must be non-degenerate. \square

Following the notation of Lemma 5.1, we set $U' := \bigcup W_x$. Then U' is an open neighbourhood of C , the map $\pi' := \pi|_{U'} : U' \rightarrow \mathcal{O}$ is a submersion, and $\tilde{N}' := \tilde{N} \cap U'$ is the set of fibrewise critical points of π' .

The sets N_{U_x} form an open covering of \tilde{N}' and each of them is a manifold, so we deduce that \tilde{N}' is a manifold. Moreover, because every critical point is fibrewise critical, N_{W_x} contains $W_x \cap C$; it follows that \tilde{N}' contains C . The bundle $E := \ker d\pi|_{\tilde{N}'}$ is complementary to $T\tilde{N}'$ in $TM|_{\tilde{N}'}$, and satisfies Condition (1') of Remark 2.2; it follows that \tilde{N}' satisfies Condition (1) of Definition 2.1. Moreover, when the compatibly fibrating neighbourhood is fibrewise orientable, the vertical tangent bundle $\ker d\pi|_{\tilde{N}'}$ is by definition orientable. But this bundle is isomorphic to the normal bundle to \tilde{N}' in M , and so \tilde{N}' is co-orientable. This completes the proof of part (a).

Since the vertical tangent bundle $\ker d\pi|_{\tilde{N}'}$ is complementary to $T\tilde{N}'$ in $TM|_{\tilde{N}'}$, there exists an open neighbourhood $U_{\tilde{N}'}$ of \tilde{N}' in U and a tubular neighbourhood map

$$\pi_{\tilde{N}'} : U_{\tilde{N}'} \rightarrow \tilde{N}'$$

whose fibres are open subsets of the fibres of π .

More precisely, let E be the pullback to \tilde{N}' of the vertical tangent bundle, that is, for each $x' \in \tilde{N}'$, the fibre of E at x' is $\ker d\pi|_{x'}$. Choose a fibrewise Riemannian metric on U' . Then the fibrewise exponential map gives a diffeomorphism from a neighbourhood of the zero section in E to a neighbourhood $U_{\tilde{N}'}$ of \tilde{N}' in M that carries the projection map to $\pi_{\tilde{N}'}$, when we identify \tilde{N}' with the zero section.

With the identification of $U_{\tilde{N}'}$ with an open subset of a vector bundle, the fibrewise second derivative of f becomes well defined on $U_{\tilde{N}'}$. The set of points where this fibrewise second derivative of f is negative definite is an open neighbourhood of \tilde{N}' in $U_{\tilde{N}'}$. We can shrink $U_{\tilde{N}'}$ so that the fibrewise second derivative is negative definite throughout $U_{\tilde{N}'}$. This, and the fact that the set of fibrewise critical points is exactly \tilde{N}' , together imply that the fibrewise maximum of f is achieved exactly at the points of \tilde{N}' .

Abusing notation by returning to our previous symbols, we now assume that we have the following set-up: \tilde{N} is a submanifold of M that contains C ; U is an open neighbourhood of \tilde{N} ; $\pi : U \rightarrow \tilde{N}$ is a tubular neighbourhood map; the set of fibrewise critical points of π is exactly \tilde{N} ; the fibrewise maxima of f are achieved exactly at the points of \tilde{N} ; the fibrewise Hessian of f is negative definite at all the points of U and, at each point x of C , the tangent space $T_x\tilde{N}$ is maximal among subspaces of T_xM on which the Hessian of f is positive semi-definite.

For every point x of C , let U_x be an open neighbourhood of x and $Z_x \subset U_x$ a minimizing submanifold for the function $f|_{U_x} : U_x \rightarrow \mathbb{R}$ and the closed subset $C \cap U_x$.

Now we want to show that f achieves its minimum value exactly on the points of C . For a point $x \in C$, let U_x be a neighbourhood for which there is a minimizing submanifold $Z_x \subset U_x$. This means that $f|_{Z_x}$ attains its minimum precisely on $C \cap Z_x$. In particular, the Hessian restricted to the tangent space TZ_x must be positive semi-definite, and hence Z_x must be transverse to the fibres of $\pi|_{U_x}$ at x itself. Therefore, $\pi|_{Z_x} : Z_x \rightarrow \tilde{N}$ is a submersion at the point $x \in Z_x$, so $\pi(Z_x)$ contains a neighbourhood of x in \tilde{N} . So $\bigcup_x \pi(Z_x)$ contains a neighbourhood of

C in \tilde{N} . Let \mathcal{O}' be such a neighbourhood. By the choice of Z_x , the restriction $f|_{Z_x}$ attains its minimum exactly on $Z_x \cap C$. Thus, for every $y \in Z_x \setminus C$, we have $f(y) > f(x)$. But we have also arranged that the fibrewise maxima of f are exactly attained on \tilde{N} . So, for every $y \in Z_x$, we have $f(y) \leq f(\pi(y))$. It follows that $f|_{\pi(Z_x)}$ attains its minimum exactly on $Z_x \cap C$. Hence, $f|_{\mathcal{O}'}$ attains its minimum exactly on C . So \mathcal{O}' is a minimizing submanifold for C , as required. This completes the proof of Proposition 3.5.

6. A MORSE-TYPE LEMMA

We now explore how minimally degenerate functions must be expressed in local coordinates. A key tool in our proof is the Morse–Bott Lemma in standard Morse–Bott theory. The complete details of the proofs of the Morse Lemma and the Morse–Bott Lemma are spelled out in [2]. We will use a parametrized version of the Morse–Bott Lemma described by Hörmander. Using that minimal degeneracy can be checked locally, we provide a description in local coordinates that is equivalent to minimal degeneracy.

Lemma 6.1. *Let M be a smooth n -dimensional manifold and $f: M \rightarrow \mathbb{R}$ a smooth function. Then f is minimally degenerate if and only if for every critical point c , there exist coordinates $x_1, \dots, x_k, y_{k+1}, \dots, y_n$ centered at c so that in a neighbourhood of c ,*

$$f = f(\mathbf{x}, \mathbf{y}) = g(\mathbf{y}) - \sum_{j=1}^k x_j^2,$$

where g is a smooth function that attains its minimum at $\mathbf{y} = 0$ and has no critical points except $\mathbf{y} = 0$.

Proof. Suppose first that there exist such coordinates near every critical point c . For a fixed critical point c , then, near c the set $N = \{ \mathbf{x} = 0 \}$ defines a minimizing submanifold for f . Thus, f satisfies minimal degeneracy, and Theorem 2.6 guarantees that f must be globally minimally degenerate.

Now suppose that f is a minimally degenerate function, and let c be a critical point. Choose coordinates $x_1, \dots, x_k, y_{k+1}, \dots, y_n$ centered at c so that $N = \{ \mathbf{x} = 0 \}$ defines a minimizing submanifold for f at c . Minimal degeneracy guarantees that the Hessian of f is negative definite in the directions normal to N . Thus, applying the Morse Lemma with Parameters (see, for example, [4, Lemma C.6.1]), we find that f has the desired form

$$f = f(\mathbf{x}, \mathbf{y}) = g(\mathbf{y}) - \sum_{j=1}^k x_j^2,$$

where g is a smooth function. We note that minimal degeneracy guarantees that g must attain its minimum at $\mathbf{y} = 0$, and after possibly shrinking the neighbourhood of c , that g has no other critical points. \square

7. NORM-SQUARE OF THE MOMENTUM MAP

Let $\Phi: M \rightarrow \mathfrak{g}^*$ be the momentum map for the action of a compact Lie group G on a symplectic manifold (M, ω) . Fix an Ad -invariant inner product on \mathfrak{g} , and fix the induced inner product on \mathfrak{g}^* .

Theorem 7.1. *The function $\|\Phi\|^2: M \rightarrow \mathbb{R}$ is minimally degenerate.*

Theorem 7.1 was proved by Kirwan in [6]. We will present here a slightly different proof: Theorem 2.6 reduces the problem to a local result, and we deduce this local result from the local normal form theorem for Hamiltonian G actions.

Remark 7.2. (1) We recall what it means for $\Phi: M \rightarrow \mathfrak{g}^*$ to be a momentum map. First, for every ξ in the Lie algebra \mathfrak{g} of G , denoting the corresponding vector field by ξ_M , we have **Hamilton's equation**:

$$(7.3) \quad d\langle \Phi, \xi \rangle = \iota_{\xi_M} \omega.$$

Second, Φ intertwines the G action on M with the coadjoint G action on the dual \mathfrak{g}^* of the Lie algebra \mathfrak{g} .

The local normal form theorem gives an explicit formula for the G -action, the symplectic form ω , and the momentum map Φ , on a neighbourhood of a G orbit.

- (2) Kirwan's motivation for generalizing Morse(–Bott) theory was in fact to apply such a theory to the norm-square of the momentum map, as suggested in the work of Atiyah and Bott [1]. In Section 8 we recall some of the consequences of this application.
- (3) Suppose that G is a torus, M is compact, and M has a G invariant Kähler metric. The compactness of M implies that the gradient flows of the components of the momentum map Φ are defined for all times, and the integrability of the complex structure implies that these flows commute. In this case, Kirwan proves in [7] that $f \circ \Phi$ is minimally degenerate for *any* convex function f ; in particular, $\|\Phi\|^2$ is minimally degenerate. So in this case one does not need the full power of Kirwan's analysis in [6] nor our local analysis here.
- (4) Notice that a connected component of the critical set of $\|\Phi\|^2$ need not be a manifold. For example, the circle action $S^1 \curvearrowright \mathbb{C}^2$ with weights 1 and -1 has momentum map

$$\begin{aligned} \Phi: \mathbb{C}^2 &\rightarrow \mathbb{R} \\ (z, w) &\mapsto \frac{|z|^2}{2} - \frac{|w|^2}{2}. \end{aligned}$$

Note that this is a Morse–Bott function, and it has a critical value at $0 \in \mathbb{R}$. The critical set for Φ is $\{0\} \subset \mathbb{C}^2$, which is a submanifold. The norm-square $\|\Phi\|^2$ also has a critical value at 0, and for the norm-square, the critical set is

$$\left\{ (z, w) \in \mathbb{C}^2 \mid \frac{|z|^2}{2} - \frac{|w|^2}{2} = 0 \right\}.$$

which is not a manifold.

We now recall some general criteria for identifying the critical set.

Lemma 7.4 (Kirwan [6, §3]). *Let $\beta = \Phi(p)$. Let T_β be the closure in G of the one parameter subgroup that is generated by the element of \mathfrak{g} that corresponds to β by the inner product. Let \mathfrak{h} denote the Lie algebra of the stabilizer of p ; let \mathfrak{h}^* be its dual, embedded in \mathfrak{g}^* by the inner product. The following conditions are equivalent.*

- (i) $p \in \text{Crit } \|\Phi\|^2$.
- (ii) $\beta \perp \text{image } d\Phi|_p$.

- (iii) $\beta \in \mathfrak{h}^*$.
- (iv) p is fixed by T_β .

Proof. Since $\|\Phi\|^2 = \langle \Phi, \Phi \rangle$, we have $d\|\Phi\|^2 = 2 \langle d\Phi, \Phi \rangle$. So $d\|\Phi\|^2|_p(v) = 2 \langle d\Phi|_p(v), \Phi(p) \rangle$. This vanishes for all $v \in T_p M$ exactly if every element in the image of $d\Phi|_p: T_p M \rightarrow \mathfrak{g}^*$ is perpendicular to $\Phi(p)$. This shows that (i) is equivalent to (ii).

The subset of \mathfrak{g}^* that is identified with \mathfrak{h}^* by the inner product is exactly the orthocomplement of the annihilator \mathfrak{h}^0 of \mathfrak{h} in \mathfrak{g}^* . But, since \mathfrak{h} is the Lie algebra of the stabilizer of p , the image of $d\Phi|_p: T_p M \rightarrow \mathfrak{g}^*$ is exactly equal to \mathfrak{h}^0 ; this is a consequence of Hamilton's equation for the momentum map and the non-degeneracy of the symplectic form ω . Thus, (ii) is equivalent to (iii).

Consider the isomorphism $\mathfrak{g}^* \xrightarrow{\cong} \mathfrak{g}$ that is induced by the inner product. Let $\widehat{\beta}$ denote the image of β . Then T_β is the closure of the one parameter subgroup generated by $\widehat{\beta}$, and so (iv) is equivalent to the condition that $\widehat{\beta}$ belong to the infinitesimal stabilizer at p . Applying the isomorphism $\mathfrak{g} \rightarrow \mathfrak{g}^*$, the relation $\widehat{\beta} \in \mathfrak{h}$ becomes (iii). \square

Example 7.5. We consider the linear action $T^2 \curvearrowright \mathbb{C}^3$ with weights $(1, 0)$, $(0, 1)$ and $(1, -1)$. That is,

$$(a, b) \cdot (z_1, z_2, z_3) = (az_1, bz_2, ab^{-1}z_3).$$

The quadratic momentum map for this action is

$$\begin{aligned} Q: \mathbb{C}^3 &\rightarrow \mathbb{R}^2 \\ (z_1, z_2, z_3) &\mapsto \left(\frac{|z_1|^2}{2} + \frac{|z_3|^2}{2}, \frac{|z_2|^2}{2} - \frac{|z_3|^2}{2} \right). \end{aligned}$$

We shift it by $(-3, 1)$, to obtain the momentum map

$$\Phi((z_1, z_2, z_3)) = \left(-3 + \frac{|z_1|^2}{2} + \frac{|z_3|^2}{2}, 1 + \frac{|z_2|^2}{2} - \frac{|z_3|^2}{2} \right).$$

The momentum image is shown in Figure 1 below.

A point $(z_1, z_2, z_3) \in \mathbb{C}^3$ is a critical point for $\|\Phi\|^2$ if and only if it satisfies one of the following conditions:

- (i) $z_1 = z_2 = z_3 = 0$;
- (ii) $z_1 = z_2 = 0$ and $\frac{|z_3|^2}{2} = 2$;
- (iii) $z_2 = z_3 = 0$ and $\frac{|z_1|^2}{2} = 3$; or
- (iv) $-3 + \frac{|z_1|^2}{2} + \frac{|z_3|^2}{2} = 0$ and $1 + \frac{|z_2|^2}{2} - \frac{|z_3|^2}{2} = 0$.

We note that Condition (i) describes a single point, and each of Conditions (ii) and (iii) describes a single one-dimensional T^2 -orbit. Condition (iii) does not define an entire Φ -level set, but each of the other conditions does. Condition (iv) defines a principal T^2 bundle over (the reduced space, which is) a two-sphere. \diamond

Let $\Phi_T: M \rightarrow \mathfrak{t}^*$ denote the momentum map for a maximal torus T of G ; thus, Φ_T is the composition of $\Phi: M \rightarrow \mathfrak{g}^*$ with the natural projection $\mathfrak{g}^* \rightarrow \mathfrak{t}^*$. Using the inner product, we can also view \mathfrak{t}^* as a *subspace* of \mathfrak{g}^* .

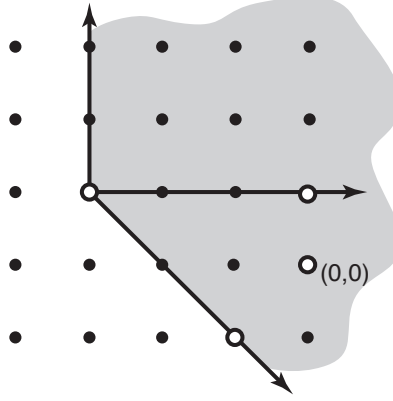


FIGURE 1. The shaded region is the momentum map image, $\Phi(\mathbb{C}^3)$. The lines are critical values for Φ , and the large dots are the critical values for $\|\Phi\|^2$.

Lemma 7.6 (Kirwan [6, Lemma 3.1]). *Suppose that $\Phi(p) \in \mathfrak{t}^*$. Then $p \in \text{Crit } \|\Phi\|^2$ if and only if $p \in \text{Crit } \|\Phi_T\|^2$.*

Proof. Let $\beta = \Phi(p)$, and let T_β denote the closure in G of the one parameter subgroup that is generated by the element of \mathfrak{g} that corresponds to β by the inner product. The assumption that $\beta \in \mathfrak{t}^*$ implies that T_β is contained in T . The lemma then follows from the equivalence of (i) and (iv) in Lemma 7.4, applied to Φ and to Φ_T . \square

We begin by proving the following special case of Theorem 7.1:

Proposition 7.7. *For a linear symplectic torus action on a symplectic vector space with quadratic momentum map Φ , the map $\|\Phi\|^2$ is minimally degenerate in a neighbourhood of the origin.*

This special case will later turn out to be relevant for the general case of a non-linear Lie group action.

Proof. Let a torus T act linearly on a symplectic vector space V . For every weight $\mu \in \mathfrak{t}_{\mathbb{Z}}^*$, let V_μ be the corresponding weight space in V . That is, denoting the corresponding character $T \rightarrow S^1$ by $a \mapsto a^\mu$, we have

$$V_\mu = \{z \in V \mid a \cdot z = a^\mu z \text{ for all } a \in T\}.$$

Then we have the weight space decomposition

$$V = \bigoplus_{\mu \in \mathcal{W}} V_\mu,$$

where $\mathcal{W} := \{\mu \in \mathfrak{t}_{\mathbb{Z}}^* \mid V_\mu \neq \{0\}\}$ is the set of weights of the representation. For $z \in V$, write

$$z = \sum_{\mu \in \mathcal{W}} z_\mu, \quad z_\mu \in V_\mu.$$

The corresponding quadratic momentum map is

$$Q: V \rightarrow \mathfrak{t}^* \quad , \quad Q(z) = \sum_{\mu \in \mathcal{W}} \frac{|z_\mu|^2}{2} \mu.$$

We will consider a momentum map whose value at the origin is not necessarily zero:

$$\Phi: V \rightarrow \mathfrak{t}^* \quad , \quad \Phi(z) = \beta + \sum_{\mu \in \mathcal{W}} \frac{|z_\mu|^2}{2} \mu,$$

for some $\beta \in \mathfrak{t}^*$. The following claim will complete the proof of Proposition 7.7.

Claim 7.8. *Let*

$$C = Q^{-1}(0) \cap \bigoplus_{\substack{\mu \text{ s.t.} \\ \langle \mu, \beta \rangle = 0}} V_\mu,$$

and let

$$N = \bigoplus_{\substack{\mu \text{ s.t.} \\ \langle \mu, \beta \rangle \geq 0}} V_\mu.$$

Then

- (1) *C is a closed subset of V and has a neighbourhood whose intersection with $\text{Crit } \|\Phi\|^2$ is exactly C.*
- (2) *N is a minimizing manifold for C.*

Proof of Part (1) of Claim 7.8. We first note that C is closed because it is the intersection of two closed subsets of V .

Now, for every subset I of \mathcal{W} , let

$$S_I = \{z \in V \mid z_\mu \neq 0 \text{ iff } \mu \in I\}.$$

Identifying the tangent spaces to V with V itself, image $d\Phi|_z$ is the same for all $z \in S_I$, and it is given by

$$\text{image } d\Phi|_z = \text{span}\{\mu \mid \mu \in I\} \quad \text{for } z \in S_I.$$

The image $\Phi(S_I)$ is an open subset of the affine space $\beta + \text{span}\{\mu \mid \mu \in I\}$. By the equivalence of (i) and (ii) in Lemma 7.4, for $z \in S_I$, we have that $z \in \text{Crit } \|\Phi\|^2$ if and only if

$$\beta + \sum_{\mu \in I} \frac{|z_\mu|^2}{2} \mu \quad \perp \quad \text{span}\{\mu \mid \mu \in I\}.$$

Thus,

$$S_I \cap \text{Crit } \|\Phi\|^2 = (\Phi|_{S_I})^{-1}(\beta'),$$

where β' is the foot of the perpendicular from the origin to the affine space

$$(\beta + \text{span}\{\mu \mid \mu \in I\}).$$

For a sufficiently small neighbourhood U of the origin in V , the neighbourhood U meets $S_I \cap \text{Crit } \|\Phi\|^2$ if and only if $\beta \perp (\text{span}\{\mu \mid \mu \in I\})$, i.e., if and only if $\beta \perp \mu$ for all $\mu \in I$. So, for a sufficiently small neighbourhood U of the origin in V , the intersection $U \cap \text{Crit } \|\Phi\|^2$ is contained in $\bigoplus_{\substack{\mu \text{ s.t.} \\ \langle \mu, \beta \rangle = 0}} V_\mu$.

Now, let

$$(7.9) \quad z \in \bigoplus_{\substack{\mu \text{ s.t.} \\ \langle \mu, \beta \rangle = 0}} V_\mu.$$

Then $z \in \text{Crit } \|\Phi\|^2$ if and only if

$$\beta + \sum \frac{|z_\mu|^2}{2} \mu \perp \text{span}\{\mu' \mid z_{\mu'} \neq 0\}.$$

By (7.9), we know that β is perpendicular to $\{\mu' \mid z_{\mu'} \neq 0\}$. So $z \in \text{Crit } \|\Phi\|^2$ if and only if

$$\sum \frac{|z_\mu|^2}{2} \mu \perp \text{span}\{\mu' \mid z_{\mu'} \neq 0\}.$$

But the vector $\sum \frac{|z_\mu|^2}{2} \mu$ is itself in the space $\text{span}\{\mu' \mid z_{\mu'} \neq 0\}$. So $z \in \text{Crit } \|\Phi\|^2$ if and only if this vector is zero. But the vector $\sum \frac{|z_\mu|^2}{2} \mu$ is exactly $Q(z)$, where $Q: V \rightarrow \mathfrak{t}^*$ is the quadratic momentum map.

We have shown that, for $z \in \bigoplus_{\substack{\mu \text{ s.t.} \\ \langle \mu, \beta \rangle = 0}} V_\mu$, we have that

$$z \in \text{Crit } \|\Phi\|^2 \text{ if and only if } Q(z) = 0.$$

This completes the proof of Part (1) of Claim 7.8. ✓

Proof of Part (2) of Claim 7.8. We have

$$\begin{aligned} \|\Phi(z)\|^2 &= \left\| \beta + \sum \frac{|z_\mu|^2}{2} \mu \right\|^2 \\ &= \|\beta\|^2 + 2 \sum_\mu \frac{|z_\mu|^2}{2} \langle \mu, \beta \rangle + \left\| \sum_\mu \frac{|z_\mu|^2}{2} \mu \right\|^2. \end{aligned}$$

Now suppose that $z \in N$. Then $z_\mu \neq 0$ only if $\langle \mu, \beta \rangle \geq 0$. From the above formula, we get that

$$\|\Phi(z)\|^2 \geq \|\beta\|^2,$$

with equality if and only if

$$\frac{|z_\mu|^2}{2} \langle \mu, \beta \rangle = 0 \text{ for all } \mu \quad \text{and} \quad \sum \frac{|z_\mu|^2}{2} \mu = 0.$$

The first of these two conditions holds if and only if

$$z \in \bigoplus_{\substack{\mu \text{ s.t.} \\ \langle \mu, \beta \rangle = 0}} V_\mu,$$

and the second holds if and only if $Q(z) = 0$. Thus, $\|\Phi(z)\|^2$ attains the minimum value $\min \|\Phi(\cdot)\|^2|_N$ if and only if $z \in C$.

Because $\|\Phi(\cdot)\|^2|_N$ attains its minimum exactly on C , the Hessian $\text{Hess } \|\Phi\|^2|_{T_x N}$ is positive semidefinite at every $x \in C$. It remains to show that, for all $x \in C$, the Hessian $\text{Hess } \|\Phi\|^2$ is negative definite on a subspace of $T_x V$ that is complementary to $T_x N$. We can take this subspace to be the image of $\bigoplus_{\substack{\mu \text{ s.t.} \\ \langle \mu, \beta \rangle < 0}} V_\mu$ under the natural

identification of V with $T_x V$. Thus, for $x \in C$, we need to show that the Hessian of the map

$$\zeta \mapsto \|\Phi(x + \zeta)\|^2$$

on the space

$$\bigoplus_{\substack{\mu \text{ s.t.} \\ \langle \mu, \beta \rangle < 0}} V_\mu$$

is negative definite at $\zeta = 0$.

For $z = x + \zeta$, write

$$(7.10) \quad \|\Phi(z)\|^2 = \|\beta\|^2 + 2 \sum_{\mu} \frac{|z_\mu|^2}{2} \langle \mu, \beta \rangle + \left\| \sum_{\mu} \frac{|z_\mu|^2}{2} \mu \right\|^2.$$

Because $x \in \bigoplus_{\substack{\mu \text{ s.t.} \\ \langle \mu, \beta \rangle = 0}} V_\mu$ and $\zeta \in \bigoplus_{\substack{\mu \text{ s.t.} \\ \langle \mu, \beta \rangle < 0}} V_\mu$, we have

$$\frac{|z_\mu|^2}{2} \langle \mu, \beta \rangle = \begin{cases} \frac{|\zeta_\mu|^2}{2} \langle \mu, \beta \rangle & \text{if } \langle \mu, \beta \rangle < 0 \\ 0 & \text{otherwise} \end{cases}$$

and

$$\begin{aligned} \sum_{\mu} \frac{|z_\mu|^2}{2} \mu &= \sum_{\substack{\mu \text{ s.t.} \\ \langle \mu, \beta \rangle = 0}} \frac{|x_\mu|^2}{2} \mu + \sum_{\substack{\mu \text{ s.t.} \\ \langle \mu, \beta \rangle < 0}} \frac{|\zeta_\mu|^2}{2} \mu \\ &= Q(x) + Q(\zeta) \\ &= Q(\zeta) \quad \text{because } Q(x) = 0. \end{aligned}$$

Substituting these in (7.10), we get

$$\|\Phi(x + \zeta)\|^2 = \text{constant} + \sum_{\substack{\mu \text{ s.t.} \\ \langle \mu, \beta \rangle < 0}} \frac{|\zeta_\mu|^2}{2} \langle \mu, \beta \rangle + \|Q(\zeta)\|^2.$$

It follows that the Hessian of $\zeta \mapsto \|\Phi(x + \zeta)\|^2$ at $\zeta = 0$ is the bilinear form that corresponds to the quadratic form

$$\zeta \mapsto \sum_{\substack{\mu \text{ s.t.} \\ \langle \mu, \beta \rangle < 0}} \frac{|\zeta_\mu|^2}{2} \langle \mu, \beta \rangle,$$

which is negative definite, as required. This completes the proof of Part (2) of Claim 7.8. \checkmark

Claim 7.8 now completes the proof of Proposition 7.7, which is the special case of Theorem 7.1 for a neighbourhood of the origin in a symplectic vector space with a linear symplectic torus action. \square

Next, we prove Theorem 7.1 for a neighbourhood of the origin in a symplectic vector space with linear symplectic *non-abelian* compact group action. The proof relies on the previous case, of a linear torus action, and will be used to prove the theorem for general non-linear actions.

Proposition 7.11. *For a linear symplectic compact group action on a symplectic vector space with a quadratic momentum map Φ , the map $\|\Phi\|^2$ is minimally degenerate in a neighbourhood of the origin.*

Proof. Consider a linear action of a (possibly non-abelian) compact Lie group H on a symplectic vector space V , with an (equivariant) momentum map

$$\Phi: V \rightarrow \mathfrak{h}^* \quad , \quad \Phi(z) = \beta + Q(z),$$

where Q is the quadratic momentum map and $\beta \in \mathfrak{h}^*$ is fixed under the coadjoint action of H .

Fix an Ad-invariant inner product on \mathfrak{h} and take the induced inner product on \mathfrak{h}^* . The inner product determines an identification of \mathfrak{h} with \mathfrak{h}^* . Let T_β denote the closure in H of the one parameter subgroup that is generated by the element of \mathfrak{h} that is identified with β by the inner product.

Fix a maximal torus T in H that contains T_β . Let

$$V = \bigoplus_{\mu \in \mathcal{W}} V_\mu$$

be the decomposition of V into weight spaces for T , and denote by z_μ the V_μ component of an element z of V .

Claim 7.12. *Let*

$$C = Q^{-1}(0) \cap \bigoplus_{\substack{\mu \text{ s.t.} \\ \langle \mu, \beta \rangle = 0}} V_\mu,$$

and let

$$N = \bigoplus_{\substack{\mu \text{ s.t.} \\ \langle \mu, \beta \rangle \geq 0}} V_\mu.$$

Then

- (1) C is closed in V and has a neighbourhood whose intersection with $\text{Crit } \|\Phi\|^2$ is exactly C .
- (2) N is a minimizing manifold for C .

Note that $\bigoplus_{\substack{\mu \text{ s.t.} \\ \langle \mu, \beta \rangle = 0}} V_\mu$ is equal to the fixed point set V^{T_β} of T_β in V .

Proof of Part (1) of Claim 7.12. We first note that C is closed because it is the intersection of two closed subsets of V .

Next, let $\Phi_T: V \rightarrow \mathfrak{t}^*$ denote the momentum map for T . Take any $\alpha \in \mathfrak{t}^*$. By our previous analysis of linear torus actions, if $\alpha \in \Phi_T(\text{Crit } \|\Phi_T\|^2)$ then there exists a subset I of \mathcal{W} such that $\alpha \perp \text{span}\{\mu' \mid \mu' \in I\}$. For each I , this condition is satisfied by only one α , namely, by the closest point to 0 in the affine space $\beta + \text{span}\{\mu' \mid \mu' \in I\}$. Thus, the set $\Phi_T(\text{Crit } \|\Phi_T\|^2)$ is finite.

Now consider $\alpha \in \mathfrak{t}^* \subset \mathfrak{h}^*$ and suppose that α is in $\Phi(\text{Crit } \|\Phi\|^2)$. Take any $z \in \text{Crit } \|\Phi\|^2$ with $\Phi(z) = \alpha$. Because $\Phi(z) \in \mathfrak{t}^*$,

- $z \in \text{Crit}(\|\Phi\|^2)$ if and only if $z \in \text{Crit}(\|\Phi_T\|^2)$ (by Lemma 7.6);
- $\Phi(z) = \Phi_T(z)$.

Thus, α is also in $\Phi_T(\text{Crit } \|\Phi_T\|^2)$. By the previous paragraph, the set of such α s is finite. So we can write

$$\Phi(\text{Crit } \|\Phi\|^2) \cap \mathfrak{t}^* = \{\beta, \alpha_1, \dots, \alpha_m\},$$

where $\alpha_1, \dots, \alpha_m$ are different from β . (The set on the left contains β because the origin in V is in $\text{Crit } \|\Phi\|^2$ and its momentum value is β .) Because $\text{Crit } \|\Phi\|^2$ is H invariant and the momentum map is H equivariant, we deduce that

$$\Phi(\text{Crit } \|\Phi\|^2) = \{\beta\} \cup \bigcup_{j=1}^m \text{Ad}^*(H)(\alpha_j).$$

Here we used that, since the momentum map is equivariant, β is fixed by the coadjoint action of H .

For each j , the coadjoint orbit $\text{Ad}^*(H)(\alpha_j)$ is closed in \mathfrak{h}^* and does not contain β , so a sufficiently small neighbourhood of β does not meet this orbit. We deduce that there exists a neighbourhood U of the origin in V such that

$$U \cap \text{Crit } \|\Phi\|^2 = \Phi^{-1}(\beta) \cap \text{Crit } \|\Phi\|^2 = Q^{-1}(0) \cap V^{T_\beta};$$

the last equality follows from Lemma 7.4. ✓

Proof of Part (2) of Claim 7.12. We proceed as in the abelian case. We write $\Phi(z) = \beta + Q(z)$, and so

$$\begin{aligned} \|\Phi(z)\|^2 &= \|\beta\|^2 + 2\langle \beta, Q(z) \rangle + \|Q(z)\|^2 \\ (7.13) \quad &= \|\beta\|^2 + 2 \sum_{\mu} \frac{|z_{\mu}|^2}{2} \langle \mu, \beta \rangle + \|Q(z)\|^2. \end{aligned}$$

For $z \in N$, this last expression is greater or equal to $\|\beta\|^2$, with equality if and only if $z \in V^{T_\beta}$ and $Q(z) = 0$. Thus, $\|\Phi\|^2|_N$ achieves its minimum precisely on C .

As before, it remains to show that the Hessian is negative definite in complementary directions. Fix an element x of C . Let's set $z = x + \zeta$, where now $\zeta \in \bigoplus_{\substack{\mu \text{ s.t.} \\ \langle \mu, \beta \rangle < 0}} V_{\mu}$.

Then

$$\frac{|z_{\mu}|^2}{2} \langle \mu, \beta \rangle = \begin{cases} \frac{|z_{\mu}|^2}{2} \langle \mu, \beta \rangle & \text{if } \langle \mu, \beta \rangle < 0 \\ 0 & \text{otherwise,} \end{cases}$$

and $Q(z) = Q(\zeta)$. Substituting this in (7.13), we get that the Hessian of $\zeta \mapsto \|\Phi(x + \zeta)\|^2$ at $\zeta = 0$ is negative definite, as required. This completes the proof of Part (2) of Claim 7.12. ✓

Claim 7.12 completes the proof of Proposition 7.11, which is the special case of Theorem 7.1 for a neighbourhood of the origin in a symplectic vector space with a linear symplectic non-abelian compact group action. □

We can now complete the proof of Theorem 7.1 in the general non-abelian non-linear case.

Proof of Theorem 7.1. Let a compact Lie group G act on a symplectic manifold M with an (equivariant) momentum map $\Phi: M \rightarrow \mathfrak{g}^*$. By the local normal form theorem (see [3, 9]; also see [11, p. 77–78]), for each G orbit $G \cdot p$ in M there exists a so-called Hamiltonian G model,

$$Y = G \times_H (V \times (\mathfrak{g}_{\beta}/\mathfrak{h})^*) \quad , \quad \Phi_Y: Y \rightarrow \mathfrak{g}^*,$$

and a G equivariant diffeomorphism from a neighbourhood of the central orbit $G \cdot [1, 0, 0]$ in Y to a neighbourhood of $G \cdot p$ in M that takes $[1, 0, 0]$ to p and whose composition with Φ is Φ_Y . Here, H is the stabilizer of p in G , \mathfrak{g}_{β} is the Lie algebra

of the stabilizer of $\beta := \Phi(p)$ under the coadjoint action, V is a symplectic vector space on which H acts linearly and symplectically with a quadratic momentum map

$$Q: V \rightarrow \mathfrak{h}^*,$$

the stabilizer H acts on $(\mathfrak{g}_\beta/\mathfrak{h})^*$ through the coadjoint representation, Y is the quotient of $G \times (V \times (\mathfrak{g}_\beta/\mathfrak{h})^*)$ by the H action

$$H \ni a: (a, z, \nu) \mapsto (ga^{-1}, a \cdot z, a \cdot \nu),$$

and, for $[g, z, \nu]$ in the model $G \times_H (V \times (\mathfrak{g}_\beta/\mathfrak{h})^*)$,

$$\Phi_Y([g, z, \nu]) = \text{Ad}^*(g)(\beta + Q(z) + \nu),$$

where \mathfrak{h}^* and $(\mathfrak{g}_\beta/\mathfrak{h})^*$ are identified with subspaces of \mathfrak{g}^* through a pre-chosen $\text{Ad}^*(G)$ -invariant inner product. We note that \mathfrak{h}^* and $(\mathfrak{g}_\beta/\mathfrak{h})^*$, as subspaces of \mathfrak{g}^* , are orthogonal to each other.

Let T_β denote the closure in H of the one parameter subgroup that is generated by the element of \mathfrak{h} that is identified with β by the inner product. Fix a maximal torus T in H that contains T_β . Let

$$V = \bigoplus_{\mu \in \mathcal{W}} V_\mu$$

be the decomposition of V into weights spaces for T .

By Theorem 2.6, we may now restrict our attention to an arbitrarily small neighbourhood of $G \cdot [1, 0, 0]$ in such a Hamiltonian G model for which $[1, 0, 0] \in \text{Crit } \|\Phi_Y\|^2$.

Claim 7.14. *Consider the subsets of V that are given by*

$$C_H = Q^{-1}(0) \cap \bigoplus_{\substack{\mu \text{ s.t.} \\ \langle \mu, \beta \rangle = 0}} V_\mu \quad \text{and} \quad N_H = \bigoplus_{\substack{\mu \text{ s.t.} \\ \langle \mu, \beta \rangle \geq 0}} V_\mu.$$

Let

$$C = G \times_H (C_H \times \{0\}),$$

and let

$$N = G \times_H (N_H \times (\mathfrak{g}_\beta/\mathfrak{g})^*).$$

Then

- (1) C is closed in the model Y , and there exists a neighbourhood of C in the model Y whose intersection with $\text{Crit } \|\Phi_Y\|^2$ is equal to C .
- (2) N is a minimizing manifold for C in the model Y .

Proof of Part (1) of Claim 7.14. By Lemma 7.4, we have that $\beta \in \mathfrak{h}^*$, and so

$$(7.15) \quad \|\Phi_Y([g, z, \nu])\|^2 = \|\beta + Q(z) + \nu\|^2 = \|\beta + Q(z)\|^2 + \|\nu\|^2.$$

The first equality is by the formula for Φ_Y and since the norm on \mathfrak{g}^* is $\text{Ad}^*(G)$ invariant. The second equality is because β and $Q(z)$ are in \mathfrak{h}^* , and ν is in $(\mathfrak{g}_\beta/\mathfrak{h})^*$, which is orthogonal to \mathfrak{h}^* .

From (7.15) we deduce that $[g, z, \nu] \in \text{Crit } \|\Phi_Y\|^2$ if and only if $\nu = 0$ and $z \in \text{Crit } \|\beta + Q(\cdot)\|^2$.

By our results for non-abelian linear group actions, there exists a neighbourhood U_H of the origin in V such that $U_H \cap \text{Crit } \|\beta + Q(\cdot)\|^2 = U_H \cap C_H$. This implies that there exists a neighbourhood U of $G \cdot [1, 0, 0]$ in Y such that $U \cap \text{Crit } \|\Phi_Y\|^2 = U \cap C$.

✓

Proof of Part (2) of Claim 7.14. Since $\|\Phi_Y([g, z, \nu])\|^2 = \|\beta + Q(z)\|^2 + \|\mu\|^2$ and $\|\beta + Q(\cdot)\|^2|_{N_H}$ attains its minimum exactly on C_H , we get that $\|\Phi_Y(\cdot)\|^2|_N$ attains its minimum exactly on C . To complete the proof, it is enough to show that, for each $[g, x, 0] \in C$, the Hessian at $\zeta = 0$ of the function

$$\bigoplus_{\substack{\mu \text{ s.t.} \\ \langle \mu, \beta \rangle < 0}} V_\mu \ni \zeta \mapsto \|\Phi_Y([g, x + \zeta, 0])\|^2$$

is negative definite. But this is exactly the function $\zeta \mapsto \|\beta + Q(x + \zeta)\|^2$, for which we have already confirmed this property. ✓

We have thus shown that the norm-square $\|\Phi\|^2$ of the momentum map for a Hamiltonian Lie group action on a symplectic manifold is locally minimally degenerate. In view of Theorem 2.6, the function $\|\Phi\|^2$ is thus globally minimally degenerate, completing the proof of Theorem 7.1. □

8. MORSE THEORETIC CONSEQUENCES

For the convenience of the reader, and to put our work in context, we now recall the main topological consequences of the fact that norm-square of the momentum map is minimally degenerate.

Let M be a compact manifold and $f: M \rightarrow \mathbb{R}$ a smooth function that is minimally degenerate. So the critical set $\text{Crit } f$ is a locally finite union of closed subsets C , on each of which f is constant, and, for each such critical set C , there exists a minimizing submanifold N_C for C .

Kirwan developed the analytic tools necessary to extend results about Morse functions to minimally degenerate functions [6, §10]. There exists a Riemannian metric on M for which the gradient vector field of f is tangent to the minimizing manifold N_C on a neighbourhood of C , for each critical set C . For such a Riemannian metric, we let

$$S_C := \left\{ x \in M \mid \begin{array}{l} \text{the gradient trajectory for } -f \\ \text{starting at } x \text{ has limit in } C \end{array} \right\}.$$

Kirwan then established the following facts. First, S_C is a submanifold of M which coincides with N_C near C . Moreover, the inclusion map $C \subset S_C$ induces an isomorphism in Čech cohomology [6, Lemma 10.17]. Finally, the submanifolds S_C give a decomposition of M into a disjoint union

$$(8.1) \quad M = \bigsqcup_C S_C$$

that satisfies the frontier condition

$$\text{closure}(S_C) \subset S_C \cup \bigcup_{\substack{C' \text{ s.t.} \\ f(C') > f(C)}} S_{C'}.$$

In the presence of a compact connected group action that preserves f , we can choose the Riemannian metric to be invariant, and then the submanifolds S_C are invariant and the inclusions $C \rightarrow S_C$ induce isomorphisms in equivariant cohomology. The decomposition (8.1) gives rise to the Morse inequalities, and, in the presence of a group action, the equivariant Morse inequalities. When $f = \|\Phi\|^2$ is the norm-square of the momentum map for a Hamiltonian action of a compact Lie group, (8.1) leads to Kirwan surjectivity [6, pp. 31–34].

Theorem 8.2 (Kirwan surjectivity). *Let a compact Lie group G on a compact symplectic manifold M with momentum map $\Phi: M \rightarrow \mathfrak{g}^*$. Then the inclusion $\Phi^{-1}(0) \rightarrow M$ induces a surjection in equivariant cohomology*

$$H_G^*(M; \mathbb{Q}) \rightarrow H_G^*(\Phi^{-1}(0); \mathbb{Q}).$$

For a sufficiently small ball U about the origin in \mathfrak{g}^* , there exists an equivariant deformation retraction from the preimage $\Phi^{-1}(U)$ to the level set $\Phi^{-1}(0)$. If 0 is a regular value of Φ , this follows from the tubular neighbourhood theorem; in general, it follows from the results of [8]. Theorem 8.2 then follows from the proof of [6, Lemma 2.18]. We note that a key technical tool in the proof of [6, Lemma 2.18] is the Atiyah–Bott Lemma [1, Proposition 13.4], which provides a condition that guarantees an equivariant Euler class to be a non-zero divisor. The Atiyah–Bott Lemma may be applied to the (normal bundles of the) strata S_C , but not necessarily to the critical sets themselves.

Remark 8.3. Lerman’s paper [8] gives a retraction from $\Phi^{-1}(U)$ to $\Phi^{-1}(0)$ which is an equivariant homotopy inverse to the inclusion map of $\Phi^{-1}(0)$ in $\Phi^{-1}(U)$. Usually this retraction is only continuous and not smooth. We believe that the inclusion map of $\Phi^{-1}(0)$ in $\Phi^{-1}(U)$ does have a smooth equivariant homotopy inverse (whose restriction to $\Phi^{-1}(0)$ is homotopic to the identity but not equal to the identity). Details will appear elsewhere.

REFERENCES

- [1] Michael F. Atiyah and Raoul Bott, *The Yang–Mills equations over Riemann Surfaces*, Phil. Trans. Royal Soc. London, A 308, 523–615 (1982).
- [2] Augustin Banyaga and David Hurtubise, *A proof of the Morse–Bott lemma*. Expo. Math. **22** (2004), no. 4, 365–373.
- [3] Victor Guillemin and Shlomo Sternberg, *A normal form for the moment map*, in: Differential geometric methods in mathematical physics (Jerusalem, 1982), 161–175, Math. Phys. Stud., 6, Reidel, Dordrecht, 1984.
- [4] Lars Hörmander, *The analysis of linear partial differential operators. III. Pseudo-differential operators*. Reprint of the 1994 edition. *Classics in Mathematics*. Springer, Berlin, 2007.
- [5] Lisa Jeffrey and Frances Kirwan, *Intersection theory on moduli spaces of holomorphic bundles of arbitrary rank on a Riemann surface*, Ann. of Math. (2) **148** (1998), no. 1, 109–196.
- [6] Frances Kirwan, *Cohomology of quotients in algebraic and symplectic geometry*, Math. Notes **31**, Princeton, 1985.
- [7] Frances Kirwan, *Some examples of minimally degenerate Morse functions*, Proc. Edinburgh Math. Soc. **30** (1987), 289–293.
- [8] Eugene Lerman, *Gradient flow of the norm squared of a moment map*, Enseign. Math. (2) **51** (2005), no. 1–2, 117–127.
- [9] Charles-Michel Marle, *Modèle d’action hamiltonienne d’un groupe de Lie sur une variété symplectique*, *Rendiconti del Seminario Matematico* **43** (1985), 227–251, Università Politecnica, Torino.
- [10] Liliu Nicolaescu, *An invitation to Morse theory*. Springer, New York, 2011.
- [11] R. Sjamaar, *Convexity properties of the momentum mapping re-examined*, Adv. Math. vol. **138**, issue 1, Sept. 1998, 46–91.
- [12] Susan Tolman and Jonathan Weitsman, *The cohomology ring of symplectic quotients*, Comm. in Anal. and Geom. **11** (2003), no. 4, 751–773.

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